Antibasis theorems for Π^0_1 classes and the jump hierarchy

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Abstract

We prove two antibasis theorems for Π^0_1 classes. The first is a jump inversion theorem for Π^0_1 classes with respect to the global structure of the Turing degrees. For any $P \subseteq 2^\omega$, define $S(P)$, the *degree spectrum* of P, to be the set of all Turing degrees $\mathbf{a}$ such that there exists $A \in P$ of degree $\mathbf{a}$. For any degree $\mathbf{a} \geq \mathbf{0}'$, let $\text{Jump}^{-1}(\mathbf{a}) = \{\mathbf{b} : \mathbf{b}' = \mathbf{a}\}$. We prove that, for any $\mathbf{a} \geq \mathbf{0}'$ and any Π^0_1 class P, if $\text{Jump}^{-1}(\mathbf{a}) \subseteq S(P)$ then P contains a member of every degree. For any degree $\mathbf{a} \geq \mathbf{0}'$ such that $\mathbf{a}$ is recursively enumerable (r.e.) in $\mathbf{0}'$, let $\text{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{0}' \text{ and } \mathbf{b}' = \mathbf{a}\}$. The second theorem concerns the degrees below $\mathbf{0}'$. We prove that for any $\mathbf{a} \geq \mathbf{0}'$ which is recursively enumerable in $\mathbf{0}'$ and any Π^0_1 class P, if $\text{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{a}) \subseteq S(P)$ then P contains a member of every degree.

Keywords: Π^0_1 classes, antibasis theorem, jump hierarchy, jump inversion.

1 Introduction

A Π^0_1 class is an effectively closed subset of the Cantor space. The study of Π^0_1 classes has led to a rich and well developed theory. Some of the most important and frequently used results are *basis* theorems: a basis theorem tells us that every nonempty Π^0_1 class has a member of a particular kind. The low basis theorem of Jockusch and Soare [9], [10], for example, tells us that every nonempty Π^0_1 class contains a member of low degree, i.e. a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{0}'$. The same authors proved that any nonempty Π^0_1 class contains a member of hyperimmune-free degree. These results are proved by the method of forcing with Π^0_1 classes in which we successively move from a set to one of its subsets in order to *force* satisfaction of a given requirement. This is a

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very general technique and can be used to obtain many useful results. Another important result by Jockusch and Soare is that every Π^0_1 class which does not contain a recursive member contains members of degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. It is possible to observe, however, that there exists a Π^0_1 class P with no recursive member such that for any $A, B \in P$ we have $\emptyset' \not\leq_T A \oplus B$, where we define $A \oplus B$ to be $\{2i : i \in A\} \cup \{2i+1 : i \in B\}$. Another example of a basis result for Π^0_1 classes is that every nonempty Π^0_1 class has a member of recursively enumerable degree. In [4], it was proven that every nonempty Π^0_1 class which does not contain a recursive member contains a member of properly low$_n$ degree, i.e. a degree $\mathbf{a}$ such that $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ but $\mathbf{a}^{(n-1)} \neq \mathbf{0}^{(n-1)}$.

An *antibasis* theorem, on the other hand, tells us that a Π^0_1 class cannot have all/any members of a particular kind without having a member of every degree. Kent and Lewis [5] proved the low antibasis theorem which says that if a Π^0_1 class contains a member of every low degree then it contains a member of every degree. We prove two antibasis theorems for Π^0_1 classes. The first concerns the global structure of the Turing degrees, and the second concerns the degrees below $\mathbf{0}'$. The proofs will be based on the jump inversion theorems in [1] and [2].

A general survey for Π^0_1 classes can be found in [4], [8].

2 Terminology and Notation

2.1 Notation

Let ω denote the set of natural numbers. We let $2^{<\omega}$ denote the set of all finite sequences of 0's and 1's. We denote sets of natural numbers with A, B, C and for a set A, $\overline{A}$ denotes the complement of A, i.e. $\omega - A$. We identify a set $A \subseteq \omega$ with its characteristic function $f : \omega \mapsto \{0, 1\}$ such that, for any $n \in \omega$, if $n \in A$ then $f(n) = 1$, and if $n \notin A$ then $f(n) = 0$. We let $\{\Psi_i\}_{i \in \omega}$ be an effective enumeration of the Turing functionals. Ψ_e is *total* if it is defined for every argument, otherwise it is called *partial*. For any $A \subseteq \omega$ and $n \in \omega$, $\Psi_e(A; n) \downarrow = m$ denotes that the e-th Turing functional with oracle A on argument n is defined and equal to m. For any A, n, $\Psi_e(A; n) \uparrow$ denotes it is not the case that $\Psi_e(A; n) \downarrow$. Since $\Psi_e(A)$ denotes a partial function and since we identify subsets of ω with their characteristic functions, it is reasonable to write $\Psi_e(A) = B$ for some $B \subseteq \omega$. We denote the Turing degrees with $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Partial functions are also denoted by f, g. We let $\langle ., . \rangle$ be a computable bijection $\omega \times \omega \to \omega$. We denote strings $\in 2^{<\omega}$ with σ, τ, ρ. We let $\sigma * \tau$ denote the concatenation of σ followed by τ. We let $\sigma \subseteq \tau$ denote that σ is an initial segment of τ. We let $\sigma \subset \tau$ mean $\sigma \subseteq \tau$ but $\sigma \neq \tau$. We say a string σ is *incompatible with* τ if neither $\sigma \subseteq \tau$ nor $\tau \subseteq \sigma$. Otherwise we say that σ is *compatible* with τ. We say that σ *extends* τ if $\tau \subseteq \sigma$. Let $|\sigma|$ denote the length of σ. $\sigma(i)$ denotes the $(i+1)$st bit of σ. For any $\sigma \in 2^{<\omega}$ and for any $n \in \omega$, we let $\Psi_e(\sigma; n)$ be defined and equal to $\Psi_e(A; n)$ if $\sigma(i) = A(i)$ for all $i < |\sigma|$ and if computing $\Psi_e(A; n)$ requires only values $A(i)$ for $i < |\sigma|$. Let $A \upharpoonright z$,

$\sigma \upharpoonright z$ denote the restriction of $A(x)$ or $\sigma(x)$ to those $x < z$. For a set $A \subseteq \omega$, we define the *jump* of A, denoted A', to be $\{e : \Psi_e(A; e) \downarrow\}$. A *tree* $T \subseteq 2^{<\omega}$ is a set of finite binary strings. We say that a set $A \subseteq \omega$ *lies on* T if there exist infinitely many $\sigma \subset A$ in T. A set A is a *path* on a tree T if A lies on T. A *leaf* of T is a string $\sigma \in T$ such that $\tau \in T$ for no $\tau \supset \sigma$. We say a tree T is *perfect* if it is nonempty and every element has at least two incompatible extensions in T. We say that σ and σ' are *e-splitting* if there exists some $n \in \omega$ such that $\Psi_e(\sigma; n) \downarrow \neq \Psi_e(\sigma'; n) \downarrow$. We say a tree T is e-splitting if every pair of incompatible strings in T is e-splitting. If $\sigma \in T$ then the *level* of σ in T is the number of proper initial segments of σ in T. If $\sigma, \tau \in T$, $\sigma \subset \tau$ and there does not exist σ' with $\sigma \subset \sigma' \subset \tau$ then we say that τ is an *immediate successor* of σ in T and σ is the *immediate predecessor* of τ in T. We let $X \subseteq 2^\omega$ be a Π^0_1 *class* if there exists a recursive predicate $\varphi(n, A)$ s.t. $A \in X \iff \forall n\, \varphi(n, A)$ where n ranges over ω and A ranges over reals. A Π^0_1 class thus can be taken as the set of infinite branches of a downward closed recursive set of finite binary strings, i.e. if $\tau \in T$ and $\sigma \subset \tau$ then $\sigma \in T$. We let $\{\Lambda_i\}_{i\in\omega}$ be an effective listing of downward closed recursive sets of strings such that for any Π^0_1 class P there exists i such that P is the set of all infinite paths through Λ_i.

2.2 Background on Π^0_1 classes

One important property of Π^0_1 classes is that for any axiomatizable theory (the deductive closure of a recursively enumerable set of sentences in a language), the set of complete and consistent extensions can be seen as a Π^0_1 class [3]. The opposite direction is also proved in [7]. That is, any Π^0_1 class can be seen as the set of complete and consistent extensions of an axiomatizable theory. Since Π^0_1 classes are defined on 2^ω, the Cantor space, it is useful to mention the compactness property of this space. This is provided by weak König's lemma which tells us that if Λ is an infinite downward closed set of finite binary strings, i.e. all initial segments of any member of the set are also in the set, then there exists an infinite path through Λ. Countable Π^0_1 classes are another type of effectively closed subset of the Cantor space. It is worth noting that countable Π^0_1 classes contain isolated points and that every isolated point is recursive [6]. So if a Π^0_1 class contains no recursive member then it must be uncountable.

3 Antibasis theorems

We begin with some definitions.

Definition 1. Let $\mathbf{E}$ be a class of Turing degrees. We say that $\mathbf{E}$ is an *antibasis* for Π^0_1 classes if whenever a Π^0_1 class contains a member of every degree $\mathbf{a} \in \mathbf{E}$, it contains a member of every degree.

Definition 2. For any $P \subseteq 2^\omega$, define $S(P)$, the *degree spectrum* of P, to be the set of all Turing degrees $\mathbf{a}$ such that there exists $A \in P$ of degree $\mathbf{a}$.

For τ which is partial computable with computable domain (possibly finite) and for every i, j, we define $\sigma(i, j, \tau)$ as follows: We let T be an i-splitting set of strings, which is recursively enumerable (in some generic fashion) and such that:

(i) all strings in T are compatible with τ;

(ii) each element which is not a leaf has precisely two immediate successors;

(iii) for any σ' which is a leaf of T there does not exist an i-splitting set of strings above σ' compatible with τ;

(iv) at each stage of the enumeration of T we only enumerate strings which properly extend leaves of the set of strings previously enumerated into T.

So roughly speaking, when τ is a finite string, T is the recursively enumerable i-splitting tree above τ. When τ has infinite domain, T is a recursively enumerable i-splitting tree in which all strings are compatible with τ. Let the strings in T be ordered according to their level and then from left to right. If there exists a string σ' in T such that either σ' is a leaf of T, or else $\Psi_i(\sigma') \notin \Lambda_j$ then define $\sigma(i, j, \tau)$ to be the least such string, where Λ_j is as defined in 2.1. If there exists no such string then $\sigma(i, j, \tau)$ is undefined. Further reading on this method can be found in [5].

Definition 3. For any degree $\mathbf{a} \geq \mathbf{0}'$, let $\text{Jump}^{-1}(\mathbf{a}) = \{\mathbf{b} : \mathbf{b}' = \mathbf{a}\}$. Similarly, for any degree $\mathbf{a} \geq \mathbf{0}'$ such that $\mathbf{a}$ is recursively enumerable (r.e.) in $\mathbf{0}'$, let $\text{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{0}' \text{ and } \mathbf{b}' = \mathbf{a}\}$.

Theorem 4. For any $\mathbf{a} \geq \mathbf{0}'$ and any Π^0_1 class P, if $\text{Jump}^{-1}(\mathbf{a}) \subseteq S(P)$ then P contains a member of every degree.

Proof. Note that if a Π^0_1 class contains all paths through a perfect computable tree, then it has a member of every degree. Given a set $A \geq_T \emptyset'$, let j be such that $[\Lambda_j] = P$ does not contain a member of every degree. Let $\sigma(i, j, \tau)$ be defined as above, for any given i, τ. Note that, since P does not have a member of every degree, $\sigma(i, j, \tau)$ is defined for all i, τ, since otherwise Λ_j is a superset of the perfect tree which is the set of all strings $\Psi_i(\tau')$ for $\tau' \in T$, with T as specifed in the definition of $\sigma(i, j, \tau)$.

We will define $B = \bigcup_{i \in \omega} \sigma_i$ such that each σ_i is finite, which is nonrecursive such that $B' \equiv_T A$ and such that if $\Psi_i(B)$ is total and nonrecursive then it is not an element of $[\Lambda_j]$ (here we do not have to consider the case that τ has infinite domain in the definition of $\sigma(i, j, \tau)$).

At stage $s = 0$, define $\sigma_0 = \emptyset$.

If $s = 4i + 1$, define $\sigma_{4i+1} = \sigma(i, j, \sigma_{4i})$.

If $s = 4i + 2$, then see if there exists $\sigma \supseteq \sigma_{4i+1}$ such that $\Psi_i(\sigma; i) \downarrow$. If so, we let $\sigma_{4i+2} = \sigma$ for smallest such σ. Otherwise just let σ_{4i+2} be some $\sigma \supseteq \sigma_{4i+1}$.

If $s = 4i + 3$, find the smallest $\sigma \supseteq \sigma_{4i+2}$ such that σ is not an initial segment of $\Psi_i(\emptyset)$. Then we let $\sigma_{4i+3} = \sigma$.

If $s = 4i + 4$, we code the i-th element of A into B simply by $\sigma_{4i+4} = \sigma_{4i+3} * \langle A(i) \rangle$.

Note that the first three steps are recursive in $\emptyset'$ which is recursive in A by hypothesis. The fourth step is recursive in A since we use it directly. Hence the construction is recursive in A. Since $i \in B' \iff \Psi_i(\sigma_{4i+2}; i) \downarrow$ we have $B' \leq_T A$. The construction is also recursive in $\emptyset' \oplus B$ since the action at stage $4i + 4$ simply adds one bit which can be determined by B. Then $i \in A$ if and only if $B(|\sigma_{4i+4}|) = 1$, so $A \leq_T \emptyset' \oplus B$. Since $B \oplus \emptyset' \leq_T B'$ we have $A \leq_T B'$. Also note that if $\Psi_e(B)$ is total and nonrecursive then it is not an element of $[\Lambda_j]$. This is satisfied at stage $4i + 1$. □

Theorem 4 basically says that for any degree $\mathbf{a} \geq \mathbf{0}'$, if a Π^0_1 class contains a member of every degree whose jump is $\mathbf{a}$ then it contains a member of every degree. We now prove the next theorem which concerns the degrees below $\mathbf{0}'$.

Theorem 5. For any $\mathbf{c} \geq \mathbf{0}'$ which is recursively enumerable in $\mathbf{0}'$ and any Π^0_1 class P, if $\text{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{c}) \subseteq S(P)$ then P contains a member of every degree.

Proof. Given a degree $\mathbf{c} \geq \mathbf{0}'$ which is r.e. in $\mathbf{0}'$, let j be such that $[\Lambda_j] = P$ does not contain a member of every degree. We aim to construct a set $A = \bigcup_{s \in \omega} \sigma_s$ by coinfinite extension such that $A \leq_T \emptyset'$ and $A' \equiv_T C$ for $C \in \mathbf{c}$ and such that $\Psi_i(A) \notin [\Lambda_j]$ for any i, if $\Psi_i(A)$ is total and non-recursive.

Let $C \in \mathbf{c}$ be r.e. in $\emptyset'$ such that $\emptyset' \leq_T C$. To satisfy $C \leq_T A'$ we want to make sure that $x \in C \iff \lim_{s \to \infty} A(\langle x, s \rangle) = 1$, so that $C \leq_T A'$ by the relativized limit lemma. Choose a one-one enumeration f of C recursive in $\emptyset'$. When a new element appears in f, we put the x-th column in A with finitely many exceptions. To make sure that $A' \leq_T C$ we will prove the existence of some function g which is recursive in C such that $\Psi_e(A; e) \downarrow$ if and only if $\Psi_e(\sigma_{g(e)}; e) \downarrow$.

At stage $s = 0$ we let $\sigma_0 = \emptyset$. At each next stage,

If $s = 3i + 1$ then $\sigma_{3i+1} = \sigma(i, j, \sigma_{3i})$. Note that we can compute this value using an oracle for $\emptyset'$ since σ_{3i} is partial computable with computable domain.

If $s = 3i + 2$ then, given σ_{3i+1}, choose some $n \in \omega$ such that $\sigma_{3i+1}(n) \uparrow$. Then define

$$\sigma_{3i+2}(n) = \begin{cases} 1 - \Psi_i(\emptyset; n) & \text{if } \Psi_i(\emptyset; n) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

If $s = 3i + 3$, given σ_{3i+2}, we look for the least $e \leq 3i + 2$ such that $\Psi_e(\sigma_{3i+2}; e) \uparrow$ and such that there exists a string σ compatible with σ_{3i+2} such that $\Psi_e(\sigma; e) \downarrow$ and giving only value 0 to elements of the columns with index smaller than e, when σ_{3i+2} is not already defined on them. If e exists, then let σ be the smallest string compatible with σ_{3i+2} and then define σ_{3i+3} as follows.

$$\sigma_{3i+3}(x) = \begin{cases} \sigma_{3i+2}(x) & \text{if } \sigma_{3i+2}(x) \downarrow \\ \sigma(x) & \text{if } \sigma(x) \downarrow \\ 1 & \text{if } x = \langle f(i), z \rangle, \text{ otherwise} \\ 0 & \text{if } x = \langle n, z \rangle \wedge n \neq f(i) \wedge n, z \leq 3i + 2 \end{cases}$$

In this case we also say that g receives attention for argument e at stage s.

If e does not exist we define σ_{3i+3} as above but we take $\sigma = \emptyset$. That is we define σ_{3i+3} in this case as

$$\sigma_{3i+3}(x) = \begin{cases} \sigma_{3i+2}(x) & \text{if } \sigma_{3i+2}(x) \downarrow \\ 1 & \text{if } x = \langle f(i), z \rangle, \text{ otherwise} \\ 0 & \text{if } x = \langle n, z \rangle \wedge n \neq f(i) \wedge n, z \leq 3i+2 \end{cases}$$

We then let $A = \bigcup_{s \in \omega} \sigma_s$. Since the construction of A is recursive in $\emptyset'$, $A \leq_T \emptyset'$ is satisfied.

Lemma 6. $C \leq_T A'$.

Proof. Since the columns that correspond to the elements of $\overline{C}$ are only finitely affected by the construction, the last clause in the definition of σ_{3i+3} ensures that A is total. We have that $A \leq_T \emptyset'$ by construction and $x \in C \iff \lim_{s \to \infty} A(\langle x, s \rangle) = 1$. So $C \leq_T A'$ is satisfied by the relativized limit lemma.

Lemma 7. $A' \leq_T C$.

Proof. We show how to construct the function g such that $\Psi_e(A; e) \downarrow$ if and only if $\Psi_e(\sigma_{g(e)}; e) \downarrow$. Choose s' large enough so that the elements smaller than e which are in C have been generated before stage s'. We can find such s' recursively in C. Then let $s'' \geq s' + 4e$ be congruent to 3 mod 4, and define $g(e) = s''$. Now we have $\Psi_e(A; e) \downarrow \iff \Psi_e(\sigma_{s''}; e) \downarrow$ since if $\Psi_e(\sigma_{s''}; e) \uparrow$ and $\Psi_e(\sigma; e) \downarrow$ for some extension σ of $\sigma_{s''}$ which is correctly defined on higher priority columns, then g would receive attention with respect to argument e at stage s''. $\square$

Corollary 8. If a Π^0_1 class contains a member of every degree of any nonrecursive jump level below $\mathbf{0}'$, then it contains a member of every degree.

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