Π^0_1 choice classes

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Abstract

We study the degree spectrum properties of so called Π^0_1 choice classes. A Π^0_1 choice class is a Π^0_1 class in which no two members have the same Turing degree. This definition leads us to some interesting cardinality properties, basis results and technically innovative constructions which might give us an insight to construct new Π^0_1 classes. The presented work can be considered as Π^0_1 choice class analogue of the work by Kent and Lewis [15].

Keywords: Π^0_1 classes, effectively closed sets, basis theorems, degree spectrum, Turing degrees, computability.

1 Notation and Terminology

We shall first give our notation and then proceed with Π^0_1 classes. We assume some familiarity with the basic properties of relative computability and Turing degrees. For a detailed account of computability, we refer the reader to [23],[6] or [10].

Let ω denote the set of natural numbers. We let $2^{<\omega}$ denote the set of all finite sequences of 0's and 1's. We denote sets of natural numbers with A, B, C and for a set A, $\overline{A}$ denotes the complement of A, i.e. $\omega - A$. The subset relation (not necessarily proper) is denoted by $\subset$. We identify a set $A \subset \omega$ with its characteristic function $f : \omega \to \{0, 1\}$ such that, for any $n \in \omega$, if $n \in A$ then $f(n) = 1$, and if $n \notin A$ then $f(n) = 0$. We let $\{\Psi_i\}_{i\in\omega}$ be an effective enumeration of the Turing functionals. Ψ_e is *total* if it is defined for every argument, otherwise it is called *partial*. The *join* of any given two sets A and B is denoted by $A \oplus B = \{2i : i \in A\} \cup \{2i+1 : i \in B\}$. For any $A \subset \omega$ and $n \in \omega$, $\Psi_e(A; n) \downarrow = m$ denotes that the e-th Turing functional with oracle A on argument n is defined and equal to m. For any A and n, $\Psi_e(A; n) \uparrow$ denotes it is not the case that $\Psi_e(A; n) \downarrow$. Since $\Psi_e(A)$ denotes a partial function and since we identify subsets of ω with their characteristic functions, it is reasonable to write $\Psi_e(A) = B$ for some $B \subset \omega$. We denote Turing degrees with boldcase letters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and we let $\mathbf{D}$ be the set of all Turing degrees. Partial functions are also denoted by f, g. We let $\langle i, j\rangle$ be a computable bijection $\omega \times \omega \to \omega$.

We denote finite strings with lowercase Greek letters like $\sigma, \tau, \eta, \rho, \pi, \upsilon$. We let $\sigma * \tau$ denote the concatenation of σ followed by τ. We let $\sigma \subset \tau$ denote that σ is an initial segment of τ. We say a string σ is *incompatible with* τ if neither $\sigma \subset \tau$ nor $\tau \subset \sigma$. Otherwise we say that σ is *compatible* with τ. Similarly, we say that σ *extends* τ if $\tau \subset \sigma$. Let $|\sigma|$ denote the length of σ. We let $\sigma(i)$ denote the $(i + 1)$st bit of σ.

For any $\sigma \in 2^{<\omega}$ and for any $n \in \omega$, we let $\Psi_e(\sigma; n)$ be defined and equal to $\Psi_e(A; n)$ if $\sigma(i) = A(i)$ for all $i < |\sigma|$ and if computing $\Psi_e(A; n)$ requires only values $A(i)$ for $i < |\sigma|$. Let $A \restriction z$ and $\sigma \restriction z$

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denote, respectively, the restriction of $A(x)$ or $\sigma(x)$ to those $x \leq z$. $\Psi_i(\sigma)[s]$ denotes $\Psi_i(\sigma)$ defined at stage s. For a set $A \subset \omega$, we define the *jump* of A, denoted A', to be $\{e : \Psi_e(A; e) \downarrow\}$.

A set T of strings is *downward closed* if $\sigma \in T$ and $\tau \subset \sigma$ then $\tau \in T$. Occasionally we refer to downward closed sets of strings as *trees*. We denote downward closed sets of strings by Λ, Υ. We say that a set A *lies on* Λ if there exist infinitely many σ in Λ such that $\sigma \subset A$. A set A is a *path* on Λ if A lies on Λ. We denote the set of infinite paths of Λ by $[\Lambda]$. A *leaf* of Λ is a string $\sigma \in \Lambda$ such that for all $\tau \supset \sigma$, $\tau \notin \Lambda$. We say that a string $\sigma \in T$ is *infinitely extendible* if there exists some $A \supset \sigma$ such that $A \in [T]$. A tree T is *perfect* if every infinitely extendible string in T has at least two incompatible extensions in T. If $\sigma \in T$ then the *level* of σ in T is the number of proper initial segments of σ in T. If $\sigma, \tau \in T$ and $\sigma \subset \tau$ and there does not exist σ' with $\sigma \subset \sigma' \subset \tau$ then we say that τ is an *immediate successor* of σ in T and σ is the *immediate predecessor* of τ in T.

We say that $\mathcal{P} \subset 2^\omega$ is a Π^0_1 *class* if there exists a downward closed computable set of strings Λ such that $\mathcal{P} = [\Lambda]$. We can then have an effective enumeration $\{\Lambda_i\}_{i\in\omega}$ of downward closed computable sets of strings such that for any Π^0_1 class $\mathcal{P}$ there exists some $i \in \omega$ such that $\mathcal{P}$ is the set of all infinite paths through Λ_i. A set $\mathcal{A} \subset 2^\omega$ is called *perfect* if there is no $f \in \mathcal{A}$ and an open set $\mathcal{O}$ such that $\mathcal{O} \cap \mathcal{A} = \{f\}$, i.e. it has no isolated points. Occasionally we use subsets of Baire space ω^ω but unless we explicitly state that, we will be working in Cantor space.

1.1 Background on Π^0_1 classes

A Π^0_1 class is an effectively closed subset in Cantor space. One important property of Π^0_1 classes is that for any computably axiomatizable theory (the deductive closure of a computably enumerable set of sentences in a language), the set of complete consistent extensions can be seen as a Π^0_1 class [21]. The opposite direction is proved in [11]. That is, any Π^0_1 class can be seen as the set of complete consistent extensions of an axiomatizable theory. The compactness property of the Cantor space is provided by the Weak König's Lemma which tells us that if Λ is an infinite downward closed set of finite strings, then there exists an infinite path through Λ.

Countable Π^0_1 classes are another type of effectively closed sets. It is important to note that countable Π^0_1 classes contain isolated points and that every isolated point is computable [16]. So if a Π^0_1 class contains no computable member then it must be uncountable.

We are particularly interested in complexity of members of Π^0_1 classes in the Turing degree universe. This has led to a rich and well developed theory. Some of the most important and frequently used results are *basis* theorems: a typical basis theorem tells us that every non-empty Π^0_1 class has a member of a particular kind. Anything which is not a basis is called *non-basis*. The *Low Basis Theorem* of Jockusch and Soare [12], [13], for example, tells us that every non-empty Π^0_1 class contains a member of low degree, i.e. a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{0}'$. The same authors proved that any non-empty Π^0_1 class contains a member of hyperimmune-free degree, i.e. a degree $\mathbf{a}$ such that for any $A \in \mathbf{a}$ and for any function $f \leq_T A$, there exists a computable function g such that $g(n) \geq f(n)$ for all n. These results are proved by the method of forcing with Π^0_1 classes in which we successively move from a set to one of its subsets in order to *force* satisfaction of a given requirement. Another important basis theorem for Π^0_1 classes is that every non-empty Π^0_1 class has a member of computably enumerable degree, i.e. the leftmost path of any downward closed computable set of strings is of c.e. degree. One interesting result by Jockusch and Soare is that every Π^0_1 class which does not contain a computable member contains members of degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. However, this does not hold for the cupping case. It is shown in [7] that there exists a Π^0_1 class $\mathcal{P}$ with no computable member such that $\emptyset' \not\leq_T A \oplus B$ for any $A, B \in \mathcal{P}$. Another non-basis result, given in [13], is that the class of computably enumerable degrees strictly below $\mathbf{0}'$ does not form a basis. Similarly, the class of computable sets does not form a basis since there exists a Π^0_1 class such that all members are non-computable. From now on we shall call Π^0_1 classes with no computable member *special* Π^0_1 classes. In [9], it was proven that every non-empty special Π^0_1 class contains a member of properly low$_n$ degree, i.e. a degree $\mathbf{a}$ such that $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ but $\mathbf{a}^{(n-1)} \neq \mathbf{0}^{(n-1)}$.

An *antibasis* theorem tells us that a Π^0_1 class cannot have all/any members of a particular kind without having a member of every degree. Kent and Lewis [15] proved the *Low Antibasis Theorem* which says that if a Π^0_1 class contains a member of every low degree then it contains a member of every degree. In [8], a stronger result was shown that for any degree $\mathbf{a} \geq \mathbf{0}'$, if a Π^0_1 class $\mathcal{P}$ contains members of every degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$, then $\mathcal{P}$ contains members of every degree. A local version of this result is also given in the same work. Namely that when $\mathbf{a}$ is also Σ^0_2, it suffices in the hypothesis to have a member of every Δ^0_2 degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$. We will give some antibasis results results for Π^0_1 choice classes and we will observe that such results become more proper when we take Π^0_1 choice classes in our hypothesis.

An extensive survey for Π^0_1 classes can be found in [2] and [9].

2 Properties of Π^0_1 choice classes

We first study the basic properties of Π^0_1 choice classes which are defined as follows.

Definition 1. A Π^0_1 class is called a *choice* class if no two members have the same Turing degree.

Definition 2 (Kent and Lewis, 2010)**.** For any $\mathcal{P} \subset 2^\omega$, define $S(\mathcal{P})$, the *degree spectrum* of $\mathcal{P}$, to be the set of all Turing degrees $\mathbf{a}$ such that there exists $A \in \mathcal{P}$ of degree $\mathbf{a}$.

Define $\mathfrak{P}_c = \{S(\mathcal{P}) : \mathcal{P} \text{ is a } \Pi^0_1 \text{ choice class}\}$. We denote the elements of $\mathfrak{P}_c$ by α, β, γ. Similarly, we define $\mathfrak{P}$ in the same manner for Π^0_1 classes. We study the structure $(\mathfrak{P}_c, <)$ where the elements are ordered by inclusion. We shall also investigate degrees which are called *choice invisible* degrees that are not contained in any of the degree spectra of Π^0_1 choice classes. This gives us proper antibasis results. Note that, for Π^0_1 classes, since they can contain members of every degree, an antibasis result for such classes makes less sense in this case. However, as we will see, one does not need to worry about this exceptional case for Π^0_1 choice classes.

It is known that Π^0_1 choice classes do exist. A strong example for a Π^0_1 choice class would be a Π^0_1 class such that every member is incomparable with each other. The existence proof was given in Theorem 4.7 of [12]. Our first observation is that if $\mathcal{P}$ is a Π^0_1 choice class then $S(\mathcal{P}) \neq \mathbf{D}$. This follows from the fact that, shown in [15], a Π^0_1 class $\mathcal{P}$ contains all paths through a perfect computable tree T iff it has members of every degree. To see why this is enough to ensure that $\mathcal{P}$ is not a Π^0_1 choice class, suppose that $\mathcal{P}$ contains all paths through T of this kind. Given any set B, we can then define a set $C_B \in [T]$ such that $C_B = \bigcup_{s\in\omega} \sigma_s$ which is of the same degree as of B. We define σ_0 to be the string at level 0 in T. Given σ_s, we let σ_{s+1} to be the leftmost successor of σ_s in T if $B(s) = 0$. Otherwise, define σ_{s+1} to be the rightmost successor of σ_s in T. Then for $B' \neq B$ but of the same degree as B, $C_{B'} \neq C_B$ but $C_{B'} \in [T]$ and $C_B \in [T]$. Note that the same argument suffices to show the other direction.

Since no Π^0_1 choice class contains members of every degree, in particular there exists a Π^0_1 class $\mathcal{P}$ such that $S(\mathcal{P}) \neq S(\mathcal{Q})$ for any Π^0_1 choice class $\mathcal{Q}$. Another interesting observation is that Π^0_1 choice classes appear to have cardinality restrictions. Clearly, a non-empty Π^0_1 choice class cannot be finite unless it has a single element, because the members of finite classes are all computable. In fact, we now show something stronger than this.

The following is a sufficient condition for the statement that every non-empty element of $\mathfrak{P}_c$, except $\{\mathbf{0}\}$, is uncountable.

Theorem 3. Any countably infinite Π^0_1 class has members of the same degree.

Proof. Let $\mathcal{P}$ be a countably infinite Π^0_1 class. We show that there are at least two computable members in $\mathcal{P}$. For this it suffices to show that in fact there are at least two isolated points. Suppose that, for the sake of contradiction, $\mathcal{P}$ is countable and has only one isolated point, say A. Let $\mathcal{Q} = \mathcal{P} - \{A\}$. Now $\mathcal{Q}$ is still a closed set because A is an isolated point. So $\mathcal{Q}$ is a Π^0_1 class which contains no isolated point. Then, $\mathcal{Q}$ is a perfect set. Therefore, $\mathcal{Q}$ is uncountable. But then, $\mathcal{P}$ is uncountable since $\mathcal{Q} \subset \mathcal{P}$. A contradiction. □

So every non-empty Π^0_1 choice class is necessarily uncountable unless it is a singleton. Since there does not exist a Π^0_1 choice class which contains members of every degree, it is natural to ask first if there exists a maximal element of $(\mathfrak{P}_c, <)$. It is known that there is no maximal element of $\mathfrak{P}$ for special Π^0_1 classes. This is provided by Jockusch and Soare [13]. The theorem says that if $\mathcal{P}$ is a special Π^0_1 class then there exists a non-zero computably enumerable degree $\mathbf{a} \notin S(\mathcal{P})$. On the other hand, for every degree $\mathbf{a}$ with $\mathbf{0} < \mathbf{a} \leq \mathbf{0}'$ there exists a special Π^0_1 class $\mathcal{P}'$ with $\mathbf{a} \in S(\mathcal{P}')$. Then $\mathcal{P}' \cup \mathcal{P}$ is a special Π^0_1 class and it properly includes $\mathcal{P}$.

We know due to Kent and Lewis [15] that for $(\mathfrak{P}, <)$, the greatest element is $\mathbf{D}$. However, since $\mathbf{D} \notin \mathfrak{P}_c$, it is reasonable to ask if there exists a maximal element in the case for Π^0_1 choice classes. We now show that there is no maximal element of $(\mathfrak{P}_c, <)$ and we do not need to worry this time about the cases where the given class contains members of every degree since the set of all Turing degrees is not a degree spectrum of a Π^0_1 choice class.

To prove that $(\mathfrak{P}_c, <)$ has no maximal element, we are given a Π^0_1 choice class $\mathcal{P}$ such that $S(\mathcal{P}) = \alpha$, where $\alpha \neq \mathbf{D}$ of course, and we want to construct a Π^0_1 choice class $\mathcal{Q} \supset \mathcal{P}$ with $S(\mathcal{Q}) = \beta$ and $\alpha < \beta$. A way to construct such $\mathcal{Q}$ is to add reals in $\mathcal{P}$ to extend it to a larger class $\mathcal{Q}$. Note that we do not need $\mathcal{Q} - \mathcal{P}$ to be infinite since it would be sufficient to add a single element whose degree is not the same as the degree of any member in $\mathcal{P}$. Other kinds of extensions are possible as well. A few notions which we are not going to cover here were introduced by Cenzer and Riazati [3] on the minimal extensions of Π^0_1 classes, and by Lawton [19] on minor superclasses of Π^0_1 classes.

We use a similar idea introduced in Cenzer and Smith [4] to construct such $\mathcal{Q}$. Now it is easy to observe that any countable set of Turing degrees is not the degree spectrum of a Π^0_1 choice class unless it is $\{\mathbf{0}\}$. The next theorem is a generalized result for adding arbitrary computably enumerable sets into a given Π^0_1 class.

Theorem 4. If α is the degree spectrum of a Π^0_1 class $\mathcal{P}$ then for any computably enumerable degree $\mathbf{a} \notin \alpha$, $\alpha \cup \{\mathbf{a}\}$ is the degree spectrum of a Π^0_1 class.

Proof. Let A be a computably enumerable set of degree $\mathbf{a}$. Suppose that we are given an enumeration of A,

$$f(n) = \mu s(A_s \upharpoonright n = A \upharpoonright n).$$

So $f(n)$ shows how long we have to wait until the enumeration of A is correct up to the initial segment of length n. Let $f_s(n) = \mu s' \leq s$ such that $A_{s'} \upharpoonright n = A_s \upharpoonright n$. Note that since f and A are both computable in each other, f is non-computable assuming that A is non-computable. However, f_s is computable. As s increases, $f_s(n)$ can only get larger.

A *copy* of $\mathcal{P} = [\Lambda]$ for some downward closed computable set of strings Λ, is defined by the set $\{\tau * \sigma : \sigma \in \Lambda\}$ for any τ. The idea behind the proof is to code the enumeration function on a path of the Π^0_1 class that we want to construct and put above some strings a copy of $\mathcal{P}$, having degree spectrum α. We shall also define a set, called Υ^*, which will be used in the construction. The role of Υ^* is to put delimeters along the way of the path of the enumeration in a way so that we can encode a kind of enumeration distance between the enumerated elements of A, i.e. the number of stages required for the enumeration of the next element in A. We keep adding a copy of $\mathcal{P}$ until we change our mind about $f_s(n)$, for some $n, s \in \omega$ and increase our guess. Clearly, $f_s(n)$ gets changed finitely many times. If we never have to come back to increase our guess about $f_s(n)$, then we are fine since we will be leaving a copy of $\mathcal{P}$ above the point where $A_s \upharpoonright n = A \upharpoonright n$. If we come back to increase our guess, we discard all but one branch and raise the delimeter symbol for coding the enumeration distance in stages. However, since there will be 0's and 1's in the copy of $\mathcal{P}$, we have to use a delimeter different than 0 or 1 as these will be coding the copy of $\mathcal{P}$. For this purpose we build our new Π^0_1 class with degree spectrum $\alpha \cup \{\mathbf{a}\}$ as a subset of $\{0, 1, 2\}^\omega$ and use the distance between 2's to code the enumeration function f.

Now suppose that $\mathcal{P} = [\Lambda]$ is a Π^0_1 class with degree spectrum α and suppose that we are given a computably enumerable degree $\mathbf{a} \notin \alpha$. We build a downward closed set of strings $\Upsilon = \bigcup_{s \in \omega} \Upsilon_s$ as a subset of $\{0,1,2\}^{<\omega}$ such that $\mathcal{Q} = [\Upsilon]$ is a Π^0_1 class with degree spectrum $\alpha \cup \{\mathbf{a}\}$. So we consider Υ like a ternary tree containing copies of $\mathcal{P}$ and a path of degree $\mathbf{a}$. When building $\mathcal{Q}$, we begin to place a copy of $\mathcal{P}$ above strings that end with a 2 in Υ_s in the form of a set of strings in Λ up to a certain length at each stage of the construction. Once the value of $f_s(n)$ is settled, we will be leaving a copy of $\mathcal{P}$. More formally, when putting the bits of Λ into Υ_s, we put Λ up to strings of length $f_s(0)+1$, $f_s(1)+1$, $f_s(2)+1$, and so on. Let us consider a set Π_s of strings of the form

$$\{0,1\}^{f_s(0)+1}2\{0,1\}^{f_s(1)+1}2\{0,1\}^{f_s(2)+1}2\cdots.$$

Since we are enumerating the branches of Λ between 2's, if we let $\Lambda \restriction n$ denote the set of strings in Λ of length n then we can put the strings in Π_s of the form

$$(\Lambda \restriction f(0)+1)2(\Lambda \restriction f(1)+1)2(\Lambda \restriction f(2)+1)2\cdots.$$

The construction is as follows.

At stage 0, we enumerate $\emptyset$ into Υ_0 and define $\Upsilon^* = \emptyset$.

At stage $s+1$. Given Π_s and Υ_s, let σ be a leaf of Υ_s.

(i) We enumerate $\sigma * d$ into Υ_{s+1} for $d \in \{0,1,2\}$ if there exists a string $\tau \in \Pi_s$ such that $\sigma * d \subset \tau$. If $d=2$, then we enumerate $\sigma * d$ also into Υ^*.

(ii) To put a copy of $\mathcal{P}$, we see if σ has an initial segment in Υ^*. If so, let $\upsilon \in \Upsilon^*$ and $\pi \in \Pi_s$ be such that $\upsilon * \pi = \sigma$. For $d \in \{0,1\}$, enumerate $\sigma * d$ into Υ_{s+1} if $\pi * d \in \Pi_s$.

As for the verification, we argue that $\mathcal{Q}$ contains a path which codes f and contains copies of $\mathcal{P}$. At stage s, let σ be a leaf of Λ_s. Whenever we find τ such that $\tau \supset \sigma * 2$, as we enumerate $\sigma * 2$ into Υ_{s+1} and Υ^*, it follows that τ with 2's removed from it is an initial segment of A_s. As s increases, so does the enumeration distance between the elements that f enumerates and coded into Υ until $f_s(n)$ is not changed anymore. To see that we leave a copy of $\mathcal{P}$ in Υ, let σ be the least string in Υ_s such that $\tau \notin \Pi_s$ for all $\tau \supset \sigma$, for a sufficiently large stage s. We know that such stage exists since every $f_s(n)$ gets changed finitely many times. Then for all $s' > s$, step (ii) guarantees by putting two incompatible extensions of τ' that $\tau * \tau' \in \Upsilon_{s'}$ for every $\tau \in \Lambda$ and hence we leave a copy of $\mathcal{P}$.

□

Corollary 5 (Cenzer and Smith, 1989)**.** For any non-zero computably enumerable degree $\mathbf{a}$, $\{\mathbf{0},\mathbf{a}\}$ is the degree spectrum of a Π^0_1 class.

We now want to show that the last theorem holds for Π^0_1 choice classes.

Theorem 6. If α is the degree spectrum of a Π^0_1 choice class $\mathcal{P}$ then for any computably enumerable degree $\mathbf{a} \notin \alpha$, $\alpha \cup \{\mathbf{a}\}$ is the degree spectrum of a Π^0_1 choice class.

Proof. Now if we want Theorem 2.4 to work for Π^0_1 choice classes we have to be careful and do some more work because we do not want to have multiple copies of the given class $\mathcal{P} = [\Lambda]$, for some downward closed computable set of strings Λ, in the class $\mathcal{Q} = [\Upsilon]$ that we want to construct. One solution is to copy mutually disjoint subclasses of $\mathcal{P}$ into different parts of $\mathcal{Q}$. However, there are some technical difficulties. If we are given a Π^0_1 choice class $\mathcal{P}$ such that $\mathcal{P} = [\Lambda]$ for some downward closed computable set of strings Λ, with a degree spectrum α and if $\mathbf{a} \notin \alpha$ is a computably enumerable degree, then we construct our new Π^0_1 choice class $\mathcal{Q}$ having degree spectrum $\alpha \cup \{\mathbf{a}\}$ in the following way.

Since we want to enumerate mutually disjoint subclasses of $\mathcal{P}$, above various strings in Υ, we have to decide which sections of $\mathcal{P}$ we should take. For this we shall define an approximate to a sequence of pairwise mutually incompatible strings $\{\sigma_s\}_{s\in\omega}$ in Λ.

For $i \in \omega$, let $[\sigma_i]$ denote the set of infinite branches of $\{\tau \in \Lambda : \tau \text{ is compatible with } \sigma_i\}$. Each σ_s will satisfy $\mathcal{P} \cap [\sigma_s] \neq \emptyset$ and if A is the leftmost path in $[\Lambda]$ then we should have that $\mathcal{P} = \{A\} \cup \bigcup_{i\in\omega}([\sigma_i] \cap \mathcal{P})$. For any $s \in \omega$, let us denote the class $[\sigma_s]$ by $\mathcal{P}^s$. Now for any $s \in \omega$, $\mathcal{P}^s$ is a Π^0_1 choice class since $\mathcal{P}^s \subset \mathcal{P}$ and also for any $s,t \in \omega$, $\mathcal{P}^s \cup \mathcal{P}^t$ is a choice class because of the fact that they are mutually disjoint subclasses of $\mathcal{P}$. Instead of adding the entire class as in the earlier proof, we keep on adding the mutually disjoint subclass of $\mathcal{P}$ above different strings in Υ. Namely, we add $\mathcal{P}^s$ for every $s \in \omega$. As we keep on enumerating strings into $\mathcal{Q}$, one of the following problems might occur in $\mathcal{P}^s$.

(i) We find out that the set of infinite paths in $[\Lambda]$ above our present approximation to some σ_s is empty.

(ii) We eventually find out that the set of infinite paths above some σ_s turns out to be the whole class $\mathcal{P}$.

The reason that these might cause problems is because we have to code the enumeration function of the given set of computably enumerable degree **a** on an infinite path of $\mathcal{Q}$ and we might need to change our guess about the sequence of mutually incomparable strings. So we have to change our mind about the values σ_s, and so about the various $\mathcal{P}^s$ of which we are placing copies in $\mathcal{Q}$. Note that it is also a problem that even if we add copies of all $\mathcal{P}^s$ into $\mathcal{Q}$, we will still miss the leftmost branch $A \in \mathcal{P}$ because for any $i, j \in \omega$, σ_i is incompatible with σ_j and if one looks at Figure 2, in such kind of mutually incompatible sequence of strings for forming a sequence of mutually disjoint subclasses of $\mathcal{P}$, the leftmost path will not be covered by the mutually incompatible sequence of strings. This leftmost path, however, is of computably enumerable degree, just like **a**. Then, instead of enumerating a single computably enumerable set into $\mathcal{Q}$, we also have to enumerate the leftmost branch of $\mathcal{P}$ that we miss. But then we have to be careful about not duplicating the branches of $\mathcal{P}$ when we put copies. We can solve this by enumerating the bits of $\mathcal{P}^s$ on two computably enumerable branches in an alternating fashion. That is, since we enumerate in two computably enumerable branches, we put the bits of $\mathcal{P}^s$ into the first computably enumerable branch then enumerate $\mathcal{P}^{s+1}$ into the second, $\mathcal{P}^{s+2}$ into the first again and so on.

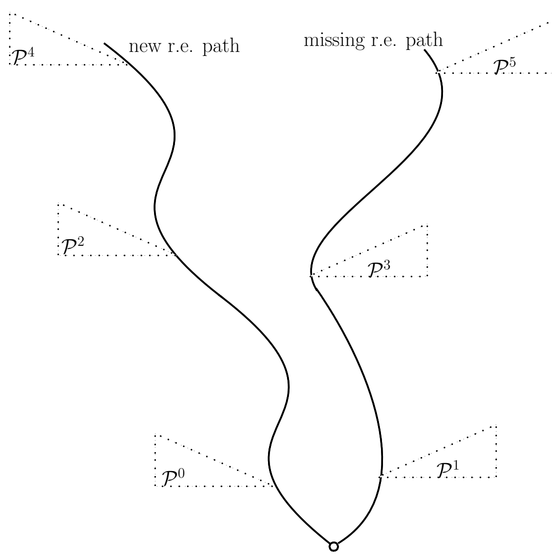

Figure 1: Two computably enumerable paths on $\mathcal{Q}$.

When we approximate the sequence $\{\sigma_s\}_{s\in\omega}$ problem (i) or (ii) may occur. To overcome these problems, it suffices to ensure that for each $i \in \omega$ the class $\mathcal{P} \cap [\sigma_i]$ is non-empty and that each branch on Λ, except the leftmost branch, extends some σ_i.

Regarding problem (ii), if there exists a string $\sigma \in \Lambda$ such that the set of infinite paths above σ is actually the entire class, then the set of infinite paths above any string $\tau \in \Lambda$ which is incompatible with σ must be empty. However, we may still have finite branches above σ. If this is the case then we have to work on the subtree above σ. If we denote the subtree of Λ above σ by Λ' and if we let

$\mathcal{P}' = [\Lambda']$ be a Π^0_1 class, clearly $\mathcal{P}'$ is a Π^0_1 choice class since $\Lambda' \subset \Lambda$. Moreover, $S(\mathcal{P}') = S(\mathcal{P})$ since $\mathcal{P}' - \mathcal{P}$ has no infinite branch, and in fact $\mathcal{P} = \mathcal{P}'$.

We shall now give the construction of the sequence of mutually incompatible strings.

Now let $\Lambda \upharpoonright n$ denote the elements of Λ of length n. We assume further that $\mathcal{P}$ has no isolated members, i.e. there does not exist any finite σ such that $\mathcal{P}$ has precisely one element extending σ. We can assume this because we can separately enumerate in any isolated path to our new class at the very end of its construction.

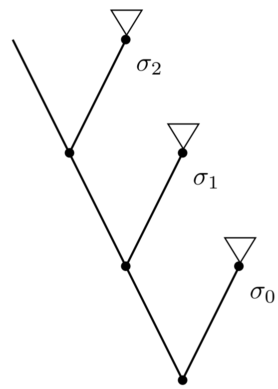

Figure 2: An illustration of how $\{\sigma_i\}_{i\in\omega}$ could be formed.

Let A be the leftmost element of $\mathcal{P}$. The following construction produces an approximation to a sequence $\{\sigma_i\}_{i\in\omega}$ such that the members of this sequence are pairwise incompatible and satisfy:

a) For each $i \in \omega$, $\mathcal{P} \cap [\sigma_i]$ is non-empty.

b) For all $B \in \mathcal{P}$ except A, there exists $i \in \omega$ with $\sigma_i \subset B$.

We define values $\sigma_i[s]$ for a finite number of i at each stage s of the construction. So $\sigma_i[s]$ shows our guess for σ_i at stage s. For each i we shall ensure $\sigma_i[s]$ is defined and takes the same value for all sufficiently large s.

At stage $s = 0$, we let $\tau = \emptyset$.

At stage $s > 0$, let τ be the leftmost element of Λ of length s. Perform the following iteration until instructed to stop:

Step i. Let ρ_i be the rightmost element of $\Lambda \upharpoonright s$ which does not extend any $\sigma_j[s]$ with $0 \le j < i$. If $\rho_i = \tau$ then terminate the iteration, and proceed to the next stage of the construction. Otherwise, let υ_i be the longest string which is an initial segment of both τ and ρ_i. Define $\sigma_i[s] = \upsilon_i * 1$, and proceed to step $i + 1$ of the iteration.

Now we verify that the sequence $\{\sigma_i[s]\}_{s\in\omega}$ converges for every $i \in \omega$ and satisfies the desired properties in (a) and (b) written above. Recall that A is the leftmost member of $\mathcal{P}$. Let B_0 be the rightmost element of $\mathcal{P}$, and let υ_0 be the longest string which is an initial segment of both A and B_0. Given B_i and υ_i, let B_{i+1} be the rightmost element of $\mathcal{P}$ extending $\upsilon_i * 0$, and let υ_{i+1} be the longest string which is an initial segment of both A and B_{i+1}.

For each i we wish to show:

(a) For all sufficiently large s we have:

$\sigma_i[s] \downarrow = \upsilon_i * 1$.

(b) All elements of $\mathcal{P}$ extending σ_i or to the right of σ_i, extend some σ_j for $j \le i$.

Suppose (a) and (b) are true for all $j < i$, and let s be large enough that, for all $s' \ge s$ and all $j < i$, $\sigma_j[s'] \downarrow = \upsilon_j * 1$.

Now let $s' > s$ be sufficiently large that there do not exist any elements of $\Lambda \upharpoonright s'$ strictly to the right of υ_i, other than those which extend some σ_j for $j < i$ (the fact that such an s' exists follows from the compactness of Cantor space, i.e. König's Lemma).

Then at all stages $s'' \geq s'$ we have $\sigma_i[s''] \downarrow = \upsilon_i * 1$, and (b) also clearly holds as required.

Now that we have $\{\sigma_i\}_{i\in\omega}$, we describe how to construct $\mathcal{Q}$. The construction of $\mathcal{Q}$ uses the previous lemma but modified as described here. For the construction we shall have a supermodule μ which handles two submodules; one for the new computably enumerable branch and one for the missing computably enumerable branch. Let us call them κ and λ, respectively. Now κ will use Theorem 2.4 but instead we put $\mathcal{P}_i$ such that $i = 2j$ for every $j \in \omega$ above the j-th enumeration point. Then, in the limit, we obtain on this side a Π^0_1 choice class with a computably enumerable branch of degree $\mathbf{a}$ with single copy of each $\mathcal{P}_i$ such that $i = 2j$ for every $j \in \omega$. Module λ is defined similarly for the missing leftmost path of computably enumerable degree and subclasses $\mathcal{P}_i$ such that $i = 2j+1$ for every $j \in \omega$. Again, we eventually obtain a Π^0_1 choice class containing a member of degree $\deg(A)$ with single copy of each $\mathcal{P}_i$ such that $i = 2j+1$ for every $j \in \omega$. The supermodule μ passes the control to κ at even stages and to λ at odd stages to fully obtain $\mathcal{Q}$. Then, $\mathcal{Q}$ is clearly a Π^0_1 choice class such that $S(\mathcal{Q}) = \alpha \cup \{\mathbf{a}\}$.

□

The proof can be easily modified to get the same result for Δ^0_2 degrees. Instead of coding the modulus function for computably enumerable sets, we code the modulus function for Δ^0_2 sets and the construction becomes very similar. Then, since a Π^0_1 choice class cannot contain members of every Δ^0_2 degree, this makes sure that the following corollaries hold.

Corollary 7. $(\mathfrak{P}_c, <)$ has no maximal element.

Definition 8. We say that β is a *minimal cover* for α if there is no $\gamma \in \mathfrak{P}_c$ strictly between α and β.

Corollary 9. For every $\alpha \in \mathfrak{P}_c$ such that $\alpha \neq \{\mathbf{0}\}$, there exists a minimal cover for α in $\mathfrak{P}_c$.

Determining, on the other hand, whether or not $\{\mathbf{0}\}$ has a minimal cover would be interesting in its own right. The following is another observation about the structure of the degree spectra of Π^0_1 choice classes.

Theorem 10. (i) $(\mathfrak{P}_c, <)$ has a least element and it is defined as $\mathbf{0}_{\mathfrak{P}_c} = \mathbf{0}_{\mathfrak{P}} = \emptyset$.

(ii) We say that $\alpha > \mathbf{0}_{\mathfrak{P}_c}$ in $\mathfrak{P}_c$ is *minimal* if there does not exist $\beta \in \mathfrak{P}_c$ with $\mathbf{0}_{\mathfrak{P}_c} < \beta < \alpha$. Then, $(\mathfrak{P}_c, <)$ has only one minimal element, i.e. $\{\mathbf{0}\}$.

Proof. There is nothing to prove for (i).

We prove (ii). Obviously $\{\mathbf{0}\}$ is minimal. Suppose that there is another minimal element of $\mathfrak{P}_c$, say α. Then there would be a Π^0_1 choice class $\mathcal{P}$ such that $S(\mathcal{P}) = \alpha$ and $\mathcal{P} = [\Lambda]$ for some downward closed computable set of strings Λ. Note that $S(\mathcal{P})$ must be uncountable. Take two immediate incompatible extensions, σ and τ, of any element of Λ. Remove every extension of τ and let $\mathcal{R}$ be the resulting class with a degree spectrum β. Now, $\mathcal{R}$ is a Π^0_1 choice class such that $\mathcal{R} \subset \mathcal{P}$ and hence $\beta < \alpha$. A contradiction. □

Definition 11. We say that a poset P has the *meet property* if for any a there exists some b such that $a \wedge b$ gives the least element of P.

We now want to show that $(\mathfrak{P}_c, <)$ has the meet property. This almost follows from a theorem due to Cole and Simpson [5]. However, to get the desired result we need to modify it for Π^0_1 choice classes. The original theorem is as follows and the proof can be found in [15].

Theorem 12 (Cole and Simpson, 2007)**.** For any special Π^0_1 class $\mathcal{P}_0$ there exists a special Π^0_1 class $\mathcal{P}_1$ such that no member of $\mathcal{P}_1$ computes any member of $\mathcal{P}_0$.

We now prove a similar statement for Π^0_1 choice classes as follows.

Theorem 13. For any Π^0_1 class $\mathcal{P}_0$ there exists a special Π^0_1 choice class $\mathcal{P}_1$ such that no member of $\mathcal{P}_1$ is Turing equivalent to a member of $\mathcal{P}_0$.

Proof. Let $\mathcal{P}_0$ be given such that $\mathcal{P}_0 = [\Lambda]$ for some downward closed computable set of strings Λ. We define an approximation to a set of strings T such that $\mathcal{P}_1 = [T]$ is a Π^0_1 choice class which satisfies the statement of the theorem. For each level of T, we aim to satisfy a single requirement for those strings at that level. Specifically, all those strings at level $3i+1$ will be defined so as to satisfy

Ξ_i : If $A \in \mathcal{P}_1$ and $\Psi_i(A)$ is total then $\Psi_i(A) \notin \mathcal{P}_0$.

For those strings at level $3i+2$, we should aim to satisfy the choiceness property (in fact we satisfy something stronger in the construction). That is,

Θ_i : If $A \in \mathcal{P}_1$ and $C \in \mathcal{P}_1$ then $A \neq \Psi_i(C)$ or $C \neq \Psi_i(A)$.

Finally, we have the non-recursiveness requirement for strings at level $3i+3$ as

Ω_i: If $A \in \mathcal{P}_1$, then $A \neq \Psi_i(\emptyset)$.

At stage $s = 0$, enumerate $\emptyset$ into T.

At stage $s > 0$,

(i) Find the least string $\tau \in T$ such that τ is of level $3i+1$, $\Psi_i(\tau)[s]$ is compatible with some string in Λ of length s and there is some leaf τ' of T extending τ such that $\Psi_i(\tau')[s]$ properly extends $\Psi_i(\tau)[s]$. If this is the case then we remove all strings extending τ from T except τ'.

(ii) We find the least string $\tau \in T$ such that $\tau \subset \Psi_i(\sigma)[s]$ for some $\sigma \in T$ of level $3i+2$ which is incompatible with τ. If such τ exists, we remove all strings extending τ from T and enumerate two incompatible extensions of σ into T.

(iii) Let $\tau \in T$ be the least string that is of level $3i+3$ and $\tau \subset \Psi_i(\emptyset)[s]$. If such τ exists, let $\tau' \in T$ be the successor of the predecessor of τ that is incompatible with τ. We then remove all strings extending τ and enumerate two incompatible extensions of τ' into T.

After these instructions, choose two incompatible strings extending each leaf of T, and enumerate these strings into T.

We claim that $\mathcal{P}_1$ is a Π^0_1 class. For this we let Υ be the set of all strings which are initial segments of strings in T at any stage. We show that Υ is downward closed, computable and $[\Upsilon] = [T]$. Now Υ is computable since we enumerate in strings that only extend strings in Υ of the previous stage. Clearly, every infinitely extendible string in T is also in Υ by the definition of Υ. The opposite direction is also true. By contrapositive, suppose that σ is not infinitely extendible in Υ. Then σ must be a leaf of T in which case σ is not infinitely extendible in T since otherwise σ would be infinitely extendible in Υ. Approximation to T converges, i.e. requirements are satisfied. It is easy to see that step (iii) ensures that $\mathcal{P}_1$ has no computable members. Note that if $\mathcal{P}_0$ has a computable member A, then $B \not\leq_T A$ is automatically satisfied for any $B \in \mathcal{P}_1$ since $\mathcal{P}_1$ is guaranteed to be a special Π^0_1 class. So $S(\mathcal{P}_0) \cap S(\mathcal{P}_1) = \emptyset$ holds when $\mathcal{P}_0$ is infinite and non-special. It is also clear that step (ii) ensures that no branch of $\mathcal{P}_1$ computes another for the choiceness property. For this suppose that $A, C \in \mathcal{P}_1$ such that $A \neq C$ and $A \equiv_T C$. Then, for all $\sigma \subset A$, where $\sigma \subset \Psi_i(C)$, we have $\sigma \in T$. Let σ_0 be the immediate predecessor of σ. Then, any extension of σ_0, compatible with σ, are not enumerated into T. But then there are finitely many σ's in T satisfying $\sigma \subset \Psi_i(C)$. Similar argument also gives a verification for the Ω_i requirements. For the Ξ_i requirements, suppose that for some least i there is a sequence $\{\tau_j\}_{j\in\omega}$ of strings such that each τ_j is a string of level $3i+1$ in T at some stage of the construction and $\tau_j \subset \tau_{j+1}$ for all j. Let $A = \bigcup_{j\in\omega} \tau_j$. Then $\Psi_i(A)$ is computable and is in $\mathcal{P}_0$. A contradiction. □

Corollary 14. $(\mathfrak{P}_c, <)$ has the meet property.

3 Decidability of the ∃-theory of $(\mathfrak{P}_c, <)$

Next, we consider the existential (∃) theory of $(\mathfrak{P}_c, <)$ and observe that it is decidable indeed. By the ∃-theory of $(\mathfrak{P}_c, <)$, we mean the set of sentences in the first order language of partial orders that are true about the degree spectra of Π^0_1 choice classes, and that are of the form $\exists x_1 \exists x_2 \cdots \exists x_k R(x_1, \ldots, x_k)$ for some $k \in \omega$, where $R(x_1, \ldots, x_k)$ is a quantifier free expression with free variables $x_1, \ldots, x_k$.

Theorem 15. The $\exists$-theory of $(\mathfrak{P}_c, <)$ is decidable.

Proof. To prove that the $\exists$-theory of $(\mathfrak{P}_c, <)$ is decidable we use the known techniques as in [1]. We define a countable infinite *independent* sequence $\{\mathcal{P}_n\}_{n\in\omega}$ of Π^0_1 choice classes with degree spectra $\{\alpha_n\}_{n\in\omega}$, i.e. a sequence satisfying that $\alpha_k \not\subset \alpha_{k_1} \cup \cdots \cup \alpha_{k_n}$ with $k \neq k_i$ for any of the k_i's.

We begin with a Π^0_1 choice class $\mathcal{P} = [\Lambda]$ for some downward closed computable set of strings Λ such that all members in $\mathcal{P}$ are Turing incomparable. Let $\{\sigma_i\}_{i\in\omega}$ be a sequence of mutually pairwise incomparable set of finite strings in Λ the same manner in Theorem 2.6. Given any $n \in \omega$, we let $\mathcal{P}_n$ to be the Π^0_1 choice class above σ_n, i.e. the set of all infinite strings in $\mathcal{P}$ extending σ_n. Note that this is a Π^0_1 choice class because all members are still Turing incomparable since $\mathcal{P}_n \subset \mathcal{P}$. If we take any finite set $J \subset \omega$ and take $\mathcal{P}' = \bigcup_{n\in J} \mathcal{P}_n$, which is a Π^0_1 choice class since $\mathcal{P}' \subset \mathcal{P}$ and $\mathcal{P}$ contains members that are Turing incomparable, then it is easy to see that $\alpha_m \not\subset S(\mathcal{P}')$ for $m \notin J$. It remains to state that there exists an embedding from any finite partially ordered set into the structure of the degree spectra of Π^0_1 choice classes. We omit the proof of this fact since it is standard (for demonstration, the reader may refer to Lemma 17 in [7]). $\square$

4 Choice invisible degrees

Next, we want to show that there exists a degree such that no Π^0_1 choice class contains a member of that degree but can be contained in a Π^0_1 class which does not contain a member of every degree. These kinds of results are often associated with antibasis theorems. Examples of antibasis theorems can be seen in [15] and [8]. When proving antibasis theorems for Π^0_1 classes, we usually exclude the case that the given class might contain a member of every degree. Then for Π^0_1 choice classes, it is more concrete to have an antibasis result since there is no such Π^0_1 choice class at all which contains a member of every degree. This way we avoid the exception of having a Π^0_1 class containing a member of every degree.

Definition 16. A degree is called *invisible* if no Π^0_1 class contains a member of that degree unless it contains a member of every degree. A degree is *choice invisible* if no Π^0_1 choice class contains a member of that degree.

Let $\mathbf{I}$ denote the set of all invisible degrees for Π^0_1 classes and let $\mathbf{CI}$ denote the set of all choice invisible degrees. Every invisible degree is choice invisible. But we ask if the relation $\mathbf{I} \subset \mathbf{CI}$ is strict and we will show that $\mathbf{CI} - \mathbf{I}$ is indeed non-empty.

Recall that a degree is PA if it contains a set which codes a complete consistent extension of Peano Artihmetic according to some computable bijection between sentences of first order language of arithmetic and the natural numbers. Although we give a more precise definition later, let us call for now a degree Martin-Löf random (1-random) if it contains a random set. It is worth noting that every degree $\mathbf{a} \geq \mathbf{0}'$ is 1-random. They are also PA since $\mathbf{0}'$ is a PA degree and PA degrees are upward closed. Moreover, if $\mathbf{a}$ is PA and 1-random, then $\mathbf{0}' \leq \mathbf{a}$. For a detailed account of the theory of algorithmic randomness we refer the reader to [10] and [20]. We first consider hyperimmune-free PA degrees for our purpose and then we look at 1-random sets.

Definition 17. (Kent and Lewis, 2010) We say that $\alpha \neq \mathbf{0}_{\mathfrak{P}}$ is *subclass invariant* if for any Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \alpha$ and any non-empty Π^0_1 class $\mathcal{P}' \subset \mathcal{P}$, $S(\mathcal{P}') = \alpha$. We say that $\alpha \neq \mathbf{0}_{\mathfrak{P}}$ is *weakly subclass invariant* if there exists a Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \alpha$ and for any non-empty Π^0_1 class $\mathcal{P}' \subset \mathcal{P}$, $S(\mathcal{P}') = \alpha$.

Now, any α which is minimal must be subclass invariant. If α is subclass invariant, suppose that $\mathcal{P}$ is a Π^0_1 class such that $S(\mathcal{P}) = \alpha$ and suppose that $\mathcal{P}'$ is a non-empty Π^0_1 class with $S(\mathcal{P}') \subset \alpha$. Then let $\mathcal{Q} = \{0 * A : A \in \mathcal{P}\} \cup \{1 * A : A \in \mathcal{P}'\}$ be a Π^0_1 class. Note that $S(\mathcal{Q}) = \alpha$, so $\mathcal{Q}$ witnesses the fact that α is not subclass invariant which is a contradiction. So subclass invariancy is equivalent to minimality.

Theorem 18. (Kent and Lewis, 2010) Suppose that α is weakly subclass invariant. If a Π^0_1 class contains any member of any hyperimmune-free degree in α then it contains a member of every degree in α.

Then, by the hyperimmune-free basis theorem, any non-empty Π^0_1 class which contains only members of degree in α contains a member of hyperimmune-free degree in α. Hence, by the theorem above, we have the fact that α is minimal if and only if it is weakly subclass invariant.

It is known that a degree is PA if and only if it contains a $\{0,1\}$-valued DNR function, where a function $f : \omega \to \omega$ is $\{0,1\}$-*valued DNR* if $f(n) \neq \Psi_n(n)$ such that f only takes values in $\{0,1\}$. Let $\boldsymbol{r}$ be the set of all 1-random degrees and let $\boldsymbol{p}$ be the set of all PA degrees. Kent and Lewis [15] showed that both $\boldsymbol{r}$ and $\boldsymbol{p}$ are minimal in $(\mathfrak{P}, <)$. This is not the case for Π^0_1 choice classes. In fact, we claim that $\boldsymbol{r}$ and $\boldsymbol{p}$ are not in $\mathfrak{P}_c$. The reason is that if a Π^0_1 class contains a member of hyperimmune-free PA degree, then it contains a member of every PA degree. This is basically followed by the hyperimmune-free basis theorem and by the fact that any non-empty Π^0_1 class containing only $\{0,1\}$-valued DNR functions contains a member of every PA degree. The proof of the latter fact, originally proved in [22], appears in [15]. We modify that proof to get the desired result. But first we need to give a lemma.

Lemma 19. If there exists a Π^0_1 choice class which contains a member of hyperimmune-free PA degree, then there exists a non-empty Π^0_1 choice class which contains only $\{0,1\}$-valued DNR functions.

Proof. Let $\mathcal{P}$ be a Π^0_1 choice class containing a hyperimmune-free PA member A. Then there exists a set B which is $\{0,1\}$-valued DNR such that $A \equiv_{tt} B$. This means there are total Turing functionals Ψ_m and Ψ_n such that $\Psi_m(A) = B$ and $\Psi_n(B) = A$. We then let $\mathcal{Q}$ contain all sets C such that $\Psi_n(C) = D$ and $\Psi_m(D) = C$, where D is a member of $\mathcal{P}$. We then let $\mathcal{Q}'$ be the elements of $\mathcal{Q}$ which are $\{0,1\}$-valued DNR. We need to argue that $\mathcal{Q}'$ is a non-empty Π^0_1 choice class. Now an infinite string is $\{0,1\}$-valued DNR if and only if there is no finite stage at which we see that some initial segment of it is not $\{0,1\}$-valued DNR. So then, we take a downward closed and computable set of strings Λ such that $\mathcal{Q} = [\Lambda]$. To form Λ' such that $\mathcal{Q}'$ is the set of infinite paths on Λ', we enumerate Λ but whenever we see that any finite string σ is not $\{0,1\}$-valued DNR, we stop enumerating in any extensions of σ. Then let $\mathcal{Q}'$ be the set of infinite paths through Λ'. Clearly, $\mathcal{Q}'$ is a non-empty Π^0_1 choice class containing only $\{0,1\}$-valued DNR functions. □

To prove our claim we now modify the proof of Theorem 5.2 in [15].

Theorem 20. Any non-empty Π^0_1 class $\mathcal{P}$ containing only $\{0,1\}$-valued DNR functions contains a member of every PA degree. Moreover, $\mathcal{P}$ contains members of the same degree.

Proof. The proof uses forcing with Π^0_1 classes. If Λ is computable and downward closed then consider $\Psi_i(\emptyset)$ such that $\Psi_i(\emptyset; i) \downarrow= n$ if and only if there exists some $l > i$ such that $\tau(i) = n$ for all $\tau \in \Lambda$ of length l. By the uniformity of the Recursion Theorem, there exists a computable function f such that, whenever $[\Lambda_j]$ is non-empty and contains only $\{0,1\}$-valued DNR functions, there exist sets $A, B \in [\Lambda_j]$ with $A(f(j)) = 0$ and $B(f(j)) = 1$. Here one can also use Lemma 2.6 in [18].

Assume that we are given j_0 such that $[\Lambda_{j_0}] = \mathcal{P}$ is non-empty and contains only $\{0,1\}$-valued DNR functions. Let A be a $\{0,1\}$-valued DNR function. We construct $B = \bigcup_{s\in\omega} \sigma_s$ which is in $\mathcal{P}$ and is of the same degree as A. We define an infinite descending sequence $[\Lambda_{j_0}] \supset [\Lambda_{j_1}] \supset [\Lambda_{j_2}] \supset \cdots$ for approximating B in $\mathcal{P}$.

At stage 0: Define $\sigma_0 = \emptyset$.

At stage $s > 0$: Suppose that we have already decided j_{s-1} and σ_{s-1}. Suppose also that there exists $C \in \left[\Lambda_{j_{s-1}}\right]$ with $C(f(j_{s-1})) = A(s-1)$.

Using an oracle for A, we can therefore compute σ of length $f(j_{s-1}) + 1$ such that $\sigma(f(j_{s-1})) = A(s-1)$ which is an initial segment of some $C \in \left[\Lambda_{j_{s-1}}\right]$. This follows from the fact that any $\{0,1\}$-valued DNR function computes a member of any non-empty Π^0_1 class such that every member is $\{0,1\}$-valued DNR.

We then define $\sigma_s = \sigma$. Then define j_s so that $[\Lambda_{j_s}]$ is the set of all $C \in \left[\Lambda_{j_{s-1}}\right]$ which extends σ.

The fact that B computes A follows from the fact that an oracle for B allows us to retrace every step of the construction defining B.

This proves the first part. Now to show that there are two members of the same degree, suppose that $\mathcal{P} = [\Lambda]$ is a Π^0_1 class, for some downward closed computable set of strings Λ, such that $\mathcal{P}$ contains only $\{0, 1\}$-valued DNR functions. We take two incompatible strings σ_0 and σ_1 in Λ. Now since every member of the set of all infinite branches above σ_0 and σ_1 is $\{0, 1\}$-valued DNR, they both contain a member of every PA degree by the previous part. Hence, they contain members of the same degree and therefore so does $\mathcal{P}$. □

Corollary 21. $\mathbf{CI} - \mathbf{I}$ is non-empty. Moreover, $\boldsymbol{p}$ is not a subset of the degree spectrum of any Π^0_1 choice class.

Proof. It follows from Lemma 4.4 and Theorem 4.5 that hyperimmune-free PA degrees are choice invisible but not invisible. □

4.1 Random sets and Π^0_1 choice classes

Definition 22. A class $\mathcal{P} \subset 2^\omega$ is of Σ^0_1-*measure zero* if there is a computably enumerable sequence of Σ^0_1 classes $\mathcal{B}_0, \mathcal{B}_1, \ldots$ such that $\forall n(\mu(\mathcal{B}_n) < 2^{-n})$ and $\mathcal{P} \subset \bigcap_{n\in\omega} \mathcal{B}_n$. A set $B \subset \omega$ is called *1-random (Martin-Löf random)* if the class $\{B\}$ is not of Σ^0_1-measure zero.

Although Π^0_1 choice classes can contain a member of PA degree, we argue that 1-random sets are too "computationally related" to be a member of a Π^0_1 choice class. The following result can be found in [14].

Theorem 23 (Kautz, 1991)**.** If a Π^0_1 class contains a 1-random set, then it is of positive measure.

The next theorem was shown by Kučera [17].

Theorem 24 (Kučera, 1985)**.** If a Π^0_1 class is of positive measure then it contains a member of every 1-random degree.

The following result shows that Π^0_1 choice classes do not contain random sets.

Theorem 25. No Π^0_1 choice class contains a 1-random set.

Proof. Suppose that a Π^0_1 class $\mathcal{P} = [\Lambda]$, for some downward closed computable set of strings Λ, contains a 1-random set. Then $\mathcal{P}$ must contain a member of every 1-random degree. Let $A, B \in \mathcal{P}$ be two 1-random sets of different degrees. Similar to Theorem 4.5, let $\sigma_0 \subset A$ and $\sigma_1 \subset B$ be two incompatible strings in Λ. Since $A \in [\sigma_0]$ and $B \in [\sigma_1]$, both $[\sigma_0]$ and $[\sigma_1]$ contain members of every 1-random degree. Therefore, $\mathcal{P}$ must contain members of the same degree. □

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